SINGULARITIES WITH THE HIGHEST MATHER MINIMAL LOG DISCREPANCY

SHIHOKO ISHII AND ANA J. REGUERA

Abstract. This paper characterizes singularities with Mather minimal log discrepancies in the highest unit interval, *i.e.,* the interval between $d-1$ and d, where d is the dimension of the scheme. The class of these singularities coincides with one of the classes of (1) compound Du Val singularities, (2) normal crossing double singularities, (3) pinch points, and (4) pairs of non-singular varieties and boundaries with multiplicities less than or equal to 1 at the point. As a corollary, we also obtain one implication of an equivalence conjectured by Shokurov for the usual minimal log discrepancies.

1. Introduction

Let (X, B) be a pair consisting of a normal variety X over an algebraically closed field k of characteristic zero and an effective $\mathbb{R}$-divisor B on X such that $K_X + B$ is an $\mathbb{R}$-Cartier divisor. The minimal log discrepancy $\mathrm{mld}(x, X, B)$ at a closed point $x \in X$ is defined for a pair and plays an important role in birational geometry. On the other hand, we can also define Mather minimal log discrepancy $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B)$ with respect to the Jacobian ideal $\mathcal{J}_X$ of X by using Mather discrepancy and the Jacobian ideal instead of the usual discrepancy (see [Is] and [DD]). Here note that we need not assume the $\mathbb{R}$-Cartier condition on $K_X + B$, and X can even be non-normal. Mather minimal log discrepancy coincides with the usual discrepancy if (X, x) is normal and locally a complete intersection. We expect Mather minimal log discrepancy also to play an important role in algebraic geometry, because it sometimes has better properties than the usual minimal log discrepancy ([Is], [DD]).

Regarding the usual minimal log discrepancy, Shokurov has proposed the following conjectures:

Conjecture 1.1 ([Sh], Conjecture 2)**.** *We have the inequality*

$$\mathrm{mld}(x; X, B) \leq \dim X,$$

where equality holds if and only if (X, x) *is non-singular and* $B = 0$ *around* x*.*

Conjecture 1.1 was proved for a non-degenerate hypersurface case ([Am2]) and a three-dimensional Gorenstein case ([Ka], [Mar]); however, it is still not proved in general. But if one replaces mld by Mather minimal log discrepancy with respect to the Jacobian ideal, then the conjecture was proved essentially in [Is], Corollary 3.15 (independently proved also in [DD], Corollary 4.15). Here we note that on a variety X (not necessarily normal), an effective $\mathbb{R}$-Cartier divisor B is defined as $B = \sum_{i=1}^{s} r_i B_i$ ($r_i \in \mathbb{R}_{\geq 0}$), where B_i is a subscheme on X defined by a principal ideal for $i = 1, \ldots, s$.

Proposition 1.2 ([Is], Corollary 3.15; [DD], Corollary 4.15)**.** *For an arbitrary variety* X *and an effective* $\mathbb{R}$*-Cartier divisor* B *on* X*, we have the inequality*

$$\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B) \leq \dim X,$$

where the equality holds if and only if (X, x) *is non-singular and* $B = 0$ *around* x*.*

Shokurov has also proposed a conjecture (not published) as follows:

Conjecture 1.3. *The inequality*

$$\dim X - 1 < \mathrm{mld}(x; X, B)$$

holds if and only if (X, x) *is non-singular and* $\mathrm{mult}_x B < 1$. *In this case, the minimal log discrepancy is computed by the exceptional divisor of the first blow-up at* x.

The implication of the "if" part of Conjecture 1.3 for the two-dimensional case was proved by Vyacheslav Shokurov in an unpublished paper, and that for the three-dimensional case was proved by Florin Ambro [Am1]; however, this conjecture is not yet proved in general. The main result of this paper is the following, which proves the Mather version of Conjecture 1.3.

Theorem 1.4. *A pair* (X, B) *consisting of an arbitrary variety* X *and an effective* $\mathbb{R}$*-Cartier divisor* B *on* X *satisfies*

$$\dim X - 1 \leq \widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B)$$

if and only if either

(i) $B = 0$, *and* (X, x) *is a normal crossing double singularity or a pinch point,*
(ii) $B = 0$, $\dim X \geq 2$ *and* (X, x) *is a compound Du Val singularity, or*
(iii) (X, x) *is non-singular and* $0 \leq \mathrm{mult}_x B \leq 1$.

In cases (i) *and* (ii), *we have* $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X) = \dim X - 1$ *and in case* (iii), *we have* $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B) = \mathrm{mld}(x; X, B) = \dim X - \mathrm{mult}_x B$, *and the minimal log discrepancy is computed by the exceptional divisor of the first blow-up at* x.

As a corollary, we obtain the "if" part of Conjecture 1.3 for the usual mld:

Corollary 1.5. *The inequality*

$$\dim X - 1 < \mathrm{mld}(x; X, B)$$

holds if (X, x) *is non-singular and* $mult_x B < 1$. *In this case the minimal log discrepancy is computed by the exceptional divisor of the first blow-up at* x.

As a further corollary, we have the following for usual mld.

Corollary 1.6. *Let* X *be locally a complete intersection at a closed point* $x \in X$. *Then,* $\mathrm{mld}(x, X, \o_X) \leq \dim X$. *Moreover,* $\mathrm{mld}(x, X, \o_X) = \dim X$ *if and only if* (X, x) *is non-singular, and* $\mathrm{mld}(x, X, \o_X) = \dim X - 1$ *if and only if* (X, x) *is either a cDV singularity, a normal crossing double singularity, or a pinch point.*

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2. Preliminaries on arcs and Mather discrepancy

2.1. Let k be a field and let X be a k-scheme. Given $m \in \mathbb{Z}_{\geq 0}$ and a field extension $k \subseteq K$, a K-arc of order m (resp. a K-arc) on X is a k-morphism $\mathrm{Spec}\, K[[t]]/(t)^{m+1} \to X$ (resp. $\mathrm{Spec}\, K[[t]] \to X$). There exists a k-scheme X_m of finite type over k, called the *space of arcs of order* m of X, whose K-rational points are the K-arcs of order m on X, for any $K \supseteq k$. There are natural affine morphisms $j_{m',m} : X_{m'} \to X_m$ for $m' > m$. The projective limit $X_\infty := \lim_{\leftarrow} X_m$ is a k-scheme (not of finite type), called the *space of arcs of* X, whose K-rational points are the K-arcs on X. We denote the natural morphisms by $j_m : X_\infty \to X_m$, $m \geq 0$.

The scheme X_∞ satisfies the following representability property: for every k-algebra A, we have a natural isomorphism

$$\mathrm{Hom}_k(\mathrm{Spec}\ A, X_\infty) \cong \mathrm{Hom}_k(\mathrm{Spec}\ A[[t]], X).$$

Given $P \in X_\infty$ with residue field $\kappa(P)$, let us denote by h_P the $\kappa(P)$-arc on X corresponding to the $\kappa(P)$-rational point of X_∞ by the previous isomorphism. The image in X of the closed point of Spec $\kappa(P)[[t]]$, or equivalently, the image of P by $j_0 : X_\infty \to X = X_0$ is called the *center* of P. Then, h_P induces a morphism of k-algebras $h_P^* : \mathcal{O}_{X,j_0(P)} \to \kappa(P)[[t]]$.

For any proper closed subset W of X, let $X_\infty^W := j_0^{-1}(W)$ (resp. $X_m^W := j_{m,0}^{-1}(W)$ for $m \geq 0$) be the closed subset of X_∞ (resp. of X_m) consisting of the arcs (resp. arcs of order m) on X, whose center lies on W. If x_0 is a fixed point in X, we simplify the previous notation setting $X_m^0 := X_m^{x_0}$ for $m \geq 0$, and $X_\infty^0 := X_\infty^{x_0}$ when there is no risk of confusion. In particular, if R is a local ring with maximal ideal M, then $(\mathrm{Spec}\ R)_m^0 := (\mathrm{Spec}\ R)_m^M$, for $m \geq 0$, and $(\mathrm{Spec}\ R)_\infty^0 := (\mathrm{Spec}\ R)_\infty^M$.

2.2. The space of arcs of the affine space $\mathbb{A}_k^N = \mathrm{Spec}\ k[x_1, \ldots, x_N]$ is

$$(\mathbb{A}_k^N)_\infty = \mathrm{Spec}\ k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots],$$

where for $n \geq 0$, $\underline{X}_n = (X_{1,n}, \ldots, X_{N,n})$ is an N-uple of variables. The space of arcs of Spec $k[[x_1, \ldots, x_N]]$ is

$$(\mathrm{Spec}\ k[[x_1, \ldots, x_N]])_\infty = \mathrm{Spec}\ k[[\underline{X}_0]][\underline{X}_1, \ldots, \underline{X}_n, \ldots].$$

For any $f \in k[x_1, \ldots, x_N]$ (resp. $f \in k[[x_1, \ldots, x_N]]$) let $\sum_{n=0}^\infty F_n\ t^n$ be the Taylor expansion of $f(\sum_n \underline{X}_n\ t^n)$, hence $F_n \in k[\underline{X}_0, \ldots, \underline{X}_n]$ (resp. $F_n \in k[[\underline{X}_0]][\underline{X}_1, \ldots, \underline{X}_n]$). If $X \subseteq \mathbb{A}_k^N$ is an affine variety, and $I_X \subset k[x_1, \ldots, x_N]$ is the ideal defining X in $\mathbb{A}_k^N$, then we have

$$X_m = \mathrm{Spec}\ k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_m]\ /\ (\{F_n\}_{0 \leq n \leq m, f \in I_X}) \quad \text{ for } m \geq 0$$
$$X_\infty = \mathrm{Spec}\ k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots]\ /\ (\{F_n\}_{n \geq 0, f \in I_X}).$$

Suppose that $\underline{0} \in X$. For $n \geq 0$, let F_n^0 denote the image of F_n by the projection $k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n] \to k[\underline{X}_1, \ldots, \underline{X}_n]$, sending $X_{i,0}$ to 0 for $1 \leq i \leq N$. Note that $F_0^0 = 0$ for $f \in I_X$. Then

$$X_m^0 = X_m^{\underline{0}} = \mathrm{Spec}\ k[\underline{X}_1, \ldots, \underline{X}_m]\ /\ (\{F_n^0\}_{1 \leq n \leq m, f \in I_X}) \quad \text{ for } m \geq 0 \tag{1}$$
$$X_\infty^0 = X_\infty^{\underline{0}} = \mathrm{Spec}\ k[\underline{X}_1, \ldots, \underline{X}_m, \ldots]\ /\ (\{F_n^0\}_{n \geq 0, f \in I_X}).$$

Analogously, if $X \subset \mathrm{Spec}\ k[[x_1, \ldots, x_N]]$ then

$$X_m = \mathrm{Spec}\ k[[\underline{X}_0]][\underline{X}_1, \ldots, \underline{X}_m]\ /\ (\{F_n\}_{0 \leq n \leq m, f \in I_X}) \quad \text{ for } m \geq 0$$
$$X_\infty = \mathrm{Spec}\ k[[\underline{X}_0]][\underline{X}_1, \ldots, \underline{X}_n, \ldots]\ /\ (\{F_n\}_{n \geq 0, f \in I_X})$$

and

$$X_m^0 = \mathrm{Spec}\ k[\underline{X}_1, \ldots, \underline{X}_m]\ /\ (\{F_n^0\}_{1 \leq n \leq m, f \in I_X}) \quad \text{ for } m \geq 0 \tag{2}$$
$$X_\infty^0 = \mathrm{Spec}\ k[\underline{X}_1, \ldots, \underline{X}_m, \ldots]\ /\ (\{F_n^0\}_{n \geq 0, f \in I_X}).$$

Here F_n^0 is the image of F_n by the projection $k[[\underline{X}_0]][\underline{X}_1, \ldots, \underline{X}_n] \to k[\underline{X}_1, \ldots, \underline{X}_n]$, sending $X_{i,0}$ to 0, for $1 \leq i \leq N$.

2.3. Given a germ of an algebraic variety (X, x_0), *i.e.,* X is a reduced separated k-scheme of finite type and x_0 is a closed point of X, recall that $X_m^0 := X_m^{x_0}$ for $m \geq 0$ and $X_\infty^0 := X_\infty^{x_0}$. Let $R = \mathcal{O}_{X,x_0}$ and M be its maximal ideal. Note that we have

$$X_m^0 \cong (\mathrm{Spec}\ R)_m^0 \quad \text{for } m \geq 0 \quad \text{and} \quad X_\infty^0 \cong (\mathrm{Spec}\ R)_\infty^0. \tag{3}$$

Let $\widehat{R}$ denote the M-adic completion of R (the notation $\widehat{\mathcal{O}_{X,x_0}}$ will also be used in text). Then, every K-arc of order m on Spec R centered at M (resp. every K-arc on Spec R centered at M) extends in a unique way to a k-morphism Spec $K[[t]]\ /\ (t)^{m+1} \to$ Spec $\widehat{R}$ (resp. Spec $K[[t]] \to$ Spec $\widehat{R}$), that is, a K-arc of order m (resp. K-arc) on Spec $\widehat{R}$, and it follows that

$$(\text{Spec } R)^0_m \cong (\text{Spec } \widehat{R})^0_m \quad \text{for } m \geq 0 \quad \text{ and } \quad (\text{Spec } R)^0_\infty \cong (\text{Spec } \widehat{R})^0_\infty. \tag{4}$$

In fact, we may suppose that $X \subseteq \mathbb{A}^N_k$ is affine, and then, applying (3), equalities (1) for $X \subseteq \mathbb{A}^N_k$ and equalities (2) for Spec $\widehat{R} \subseteq$ Spec $k[[x_1, \ldots, x_N]]$, an explicit description of the isomorphisms in (4) follows.

2.4. Henceforth, k will be an algebraically closed field of characteristic zero, and X an algebraic variety over k of dimension d. Given a resolution of the singularities $\pi : Y \to X$ and a prime divisor E on Y contained in the exceptional locus of π, recall that Y^E_∞ is the inverse image of E by the projection $Y_\infty \to Y$ and let N_E be the closure of its image $\pi_\infty(Y^E_\infty)$ by $\pi_\infty : Y_\infty \to X_\infty$, which is an irreducible subset of $X^{\text{Sing X}}_\infty$. The generic point P_E of N_E is a stable point of X_∞ (see [Re], 3.1). Therefore, $\dim \mathcal{O}_{\overline{j_m(X_\infty)}, j_m(P_E)}$ is constant for $m \gg 0$ and we have

$$\dim \mathcal{O}_{X_\infty, P_E} \leq \sup_m \dim \mathcal{O}_{\overline{j_m(X_\infty)}, j_m(P_E)} < \infty \tag{5}$$

Although the first inequality may be strict (see Example 2.8 below), the constant $\dim \mathcal{O}_{\overline{j_m(X_\infty)}, j_m(P_E)}$, for $m \gg 0$, is called the codimension of N_E in X_∞ (see [EM], sec. 5). In [DEI], the value of this constant has been described in terms of the relative Mather canonical divisor $\widehat{K}_{Y/X}$. A review of this concept is given below.

2.5. If $\pi : Y \to X$ is a resolution of the singularities dominating the Nash blowing-up of X (for the definition of Nash blowing-up, see, for example [DEI], Definition 1.1), then the image of the canonical homomorphism $d\pi : \pi^*(\wedge^d \Omega_X) \to \wedge^d \Omega_Y$ is an invertible sheaf. More precisely, there exists an effective divisor $\widehat{K}_{Y/X}$ with support in the exceptional locus of π such that

$$d\pi(\pi^*(\wedge^d \Omega_X)) = \mathcal{O}_Y(-\widehat{K}_{Y/X})\ \wedge^d \Omega_Y.$$

The divisor $\widehat{K}_{Y/X}$ is called the *Mather discrepancy divisor.*

For any prime divisor E on Y, let

$$\widehat{k}_E := \text{ord}_E(\widehat{K}_{Y/X}).$$

Here $\widehat{k}_E$ is an integer because $\widehat{K}_{Y/X}$ is a divisor on Y. Note that $\widehat{k}_E \neq 0$ implies that E is contained in the exceptional locus of π and that $\widehat{k}_E$ depends only on the divisorial valuation ν_E defined by E. That is, if $\pi : Y' \to X$ is another resolution of singularities dominating the Nash blowing-up of X such that the center of ν_E on Y' is a divisor E', then $\text{ord}_E(\widehat{K}_{Y/X}) = \text{ord}_{E'}(\widehat{K}_{Y'/X})$. In general, for every prime divisor E over X, *i.e.,* a prime divisor on a normal variety Y, which is proper birational over X, we can define $\widehat{k}_E$ because there exists a resolution Y' of the singularities that dominates both the Nash blowing-up and Y, and E appears on Y' also.

Then we have

$$\sup_m \dim \mathcal{O}_{\overline{j_m(X_\infty)}, j_m(P_E)} = \widehat{k}_E + 1 \tag{6}$$

([DEI], Theorem 3.9).

For each divisorial valuation $\nu = \nu_E$, where E is a prime divisor over X, let $c_X(\nu_E)$ denote the center of ν_E on X. Given non-zero ideal sheaves in $\mathcal{O}_X$ with non-negative powers and a closed subset W of X, *Mather minimal log discrepancy* along W is defined as follows:

Definition 2.6. Let X be a variety over k. Given non-zero ideal sheaves $\mathfrak{a}_1, \dots, \mathfrak{a}_l$ of $\mathcal{O}_X$, non-negative real numbers $s_1, \dots, s_l$, and a proper closed subset W of X, *Mather minimal log discrepancy of* $(X, \mathfrak{a}_1^{s_1}, \dots, \mathfrak{a}_l^{s_l})$ along W is defined as follows:

$$\begin{aligned}(7)\qquad &\widehat{\mathrm{mld}}(W; X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_l^{s_l}) \\ &:= \inf \{\widehat{k}_E - \sum_{i=1}^{l} s_i \mathrm{ord}_E(\mathfrak{a}_i) + 1 \mid \ \nu_E \text{ divisorial valuation}, c_X(\nu_E) \subseteq W\},\end{aligned}$$

if $\dim X \geq 2$ or $\dim X = 1$ and the infimum on the right hand side is non-negative; otherwise, $\widehat{\mathrm{mld}}(W; X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_l^{s_l}) := -\infty$. Here $\widehat{k}_E = \mathrm{ord}_E \widehat{K}_{Y/X}$ for a resolution $Y \to X$ on which E appears. A remark on this definition: if $\dim X \geq 2$, then the infimum in (7) is negative if and only if it is equal to $-\infty$ ([Is], Remark 3.4).

Let X be a variety and $B = \sum_{j=1}^{r} b_j B_j$ an effective $\mathbb{R}$-Cartier divisor, *i.e.,* $b_j \in \mathbb{R}_{\geq 0}$ and B_j is a subscheme defined by a principal ideal on X $(j = 1, \dots, r)$. For non-zero ideal sheaves $\mathfrak{a}_1, \dots, \mathfrak{a}_l$ of $\mathcal{O}_X$, positive real numbers $s_1, \dots, s_l$, and an effective $\mathbb{R}$-divisor B, we define Mather minimal log discrepancy for the mixture of ideals and a divisor $(\mathfrak{a}_1^{s_1}, \dots, \mathfrak{a}_l^{s_l}, B)$ as follows

$$\widehat{\mathrm{mld}}(W; X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_l^{s_l} \cdot B) := \widehat{\mathrm{mld}}(W; X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_l^{s_l} \cdot \mathcal{O}_X(-B_1)^{b_1} \cdots \mathcal{O}_X(-B_r)^{b_r}).$$

2.7. If X is a normal affine Gorenstein variety and $\pi : Y \to X$ is a resolution of the singularities dominating the Nash blowing-up of X, then we have

$$(8)\qquad \mathcal{O}_Y(-\widehat{K}_{Y/X}) = \pi^*(I_Z) \otimes \mathcal{O}_Y(-K_{Y/X}),$$

where $K_{Y/X}$ is the unique divisor with support in the exceptional locus of π, which is linearly equivalent to $K_Y - \pi^*(K_X)$, and I_Z is the ideal defining the first Nash subscheme of X ([EM], Appendix).

In particular, if X is normal and a complete intersection, then $I_Z = \mathcal{J}_X$ is the Jacobian ideal of X, and from (8), it follows that for any non-zero ideal sheaf $\mathfrak{a}$ of $\mathcal{O}_X$ and any proper closed subset W of X, we have

$$(9)\qquad \widehat{\mathrm{mld}}(W; X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_l^{s_l} \mathcal{J}_X) = \mathrm{mld}(W; X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_l^{s_l}),$$

where recalling that $\mathrm{mld}(W; X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_l^{s_l})$ is the minimal log discrepancy of $(X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_l^{s_l})$ along W defined as follows:

$$\begin{aligned}&\mathrm{mld}(W; X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_l^{s_l}) \\ &:= \inf \{k_E - \sum_{i=1}^{l} s_i \mathrm{ord}_E(\mathfrak{a}_i) + 1 \mid \ \nu_E \text{ divisorial valuation}, c_X(\nu_E) \subseteq W\}\end{aligned}$$

if $\dim X \geq 2$ or $\dim X = 1$ and the infimum on the right hand side is non-negative; otherwise, $\widehat{\mathrm{mld}}(W; X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_l^{s_l}) := -\infty$. Here $k_E := \mathrm{ord}_E K_{Y/X}$, where $Y \to X$ is a desingularization and E is the center of ν_E on Y and only depends on the divisorial valuation ν_E.

Example 2.8. Let X be the hypersurface $x_1^2+x_2^2+x_3^2=0$ in $\mathbb{A}_k^3$, which has an $\mathbf{A}_1$-singularity at $\underline{0}$. The exceptional locus of its minimal desingularization $\pi : Y \to X$ consists of a unique irreducible curve E. We have $\dim \mathcal{O}_{X_\infty, P_E} = 1$ ([Re] Corollaries 5.12 and 5.15). On the other hand, X has a canonical singularity at $\underline{0}$, hence $k_E = 0$ and, by (8), $\widehat{k}_E = 1$. From (6), we conclude that

$$1 = \dim \mathcal{O}_{X_\infty, P_E} < \sup_m \dim \mathcal{O}_{\overline{j_m(X_\infty)}, j_m(P_E)} = \widehat{k}_E + 1 = 2$$

That is, in this example, the first inequality in (5) is strict.

2.9. *Inversion of adjunction* ([Is], Proposition 3.10, [DD], Theorem 4.10): Let X be an algebraic variety, A a non-singular variety containing X as a closed subvariety of codimension c, and W a proper closed subset of X. Let $\widetilde{\mathfrak{a}} \subseteq \mathcal{O}_A$ be an ideal sheaf such that $\mathfrak{a} := \widetilde{\mathfrak{a}}\mathcal{O}_X$ is a non-zero ideal sheaf of $\mathcal{O}_X$, and let $I_X \subseteq \mathcal{O}_A$ be the ideal sheaf defining X. Then

$$\widehat{\mathrm{mld}}(W; X, \mathfrak{a}\mathcal{J}_X) = \widehat{\mathrm{mld}}(W; A, \widetilde{\mathfrak{a}} I_X^c). \tag{10}$$

This result is a generalization of an analogous result for minimal log discrepancies proved in [EM], Theorem 8.1.

Consider equality (10) for trivial $\mathfrak{a}, \tilde{\mathfrak{a}}$ and $W = \{x_0\}$ for a closed point x_0. By [Is], Proposition 3.7, the right hand side is represented as follows:

$$\widehat{\mathrm{mld}}(x_0; A, I_X^c) = \inf_m \{\mathrm{codim}(\mathrm{Cont}^{\geq m}(I_X) \cap \mathrm{Cont}^{\geq 1}(M_{x_0}), A_\infty) - cm\}, \tag{11}$$

where the codimension, as in the sense of [EM] (see 2.4), is the minimal value of the $\{\mathrm{codim}(C_i, A_\infty)\}$ of the irreducible components C_i in A_∞. Here if P_i is the generic point of the component C_i, then $\mathrm{codim}(C_i, A_\infty)$ is defined as $\sup_m \dim \mathcal{O}_{\overline{j_m(A_\infty)}, j_m(P_i)}$ and it coincides with the constant

$$\dim \mathcal{O}_{j_m(A_\infty), j_m(P_i)} = \dim \mathcal{O}_{A_m, j_m(P_i)} \quad \text{for } m \gg 0.$$

Here by using the canonical projection $j_r : A_\infty \to A_r$, the "contact loci" are represented as follows:

$$\mathrm{Cont}^{\geq m}(I_X) := \{P \in A_\infty \mid \mathrm{ord}_t h_P^*(I_X) \geq m\} = j_{m-1}^{-1}(X_{m-1}),$$

$$\mathrm{Cont}^{\geq 1}(M_{x_0}) := \{P \in A_\infty \mid \mathrm{ord}_t h_P^*(M_{x_0}) \geq 1\} = j_0^{-1}(x_0).$$

Therefore, (11) turns out to be

$$\widehat{\mathrm{mld}}(x_0; A, I_X^c) = \inf_m \{\mathrm{codim}(X_{m-1} \cap j_{m-1,0}^{-1}(x_0), A_{m-1}) - cm\}. \tag{12}$$

From (10), (12), and also replacing $m-1$ with m, we obtain

$$\widehat{\mathrm{mld}}(x_0; X, \mathcal{J}_X) = \inf_m \{\ (m+1)d - \dim X_m^0\ \}, \tag{13}$$

where $d = \dim X$.

Proposition 2.10. *For an arbitrary variety X of dimension d and an effective $\mathbb{R}$-Cartier divisor B on X, we have the inequality*

$$\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B) \leq d,$$

where the equality holds if and only if $B = 0$ around x and (X, x) is non-singular.

Proof: In [Is], Corollary 3.15, it was proved that

$$\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X) \leq d$$

always holds and the equality holds if and only if (X, x) is non-singular. Therefore, it is sufficient to prove that $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B) < d$ for any B non-zero around x.

And this inequality holds because $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B) < \widehat{\mathrm{mld}}(x; X, \mathcal{J}_X)$ holds for any B non-zero around x. $\square$

3. Characterization of top singularities

Definition 3.1. For $d \geq 1$, we say that a d-dimensional variety X has a *top singularity* at x_0 or that (X, x_0) is a top singularity, if $\widehat{\mathrm{mld}}(x_0; X, \mathcal{J}_X) = d - 1$.

Definition 3.2. Let (X, x_0) be a germ of a hypersurface in $\mathbb{A}_k^{d+1}$ with $d \geq 2$. A singularity (X, x_0) is called a *compound Du Val singularity* or cDV singularity if (i) it is Du Val in the case of $d = 2$, (ii) its general hyperplane section is a Du Val singularity in the case of $d = 3$, (iii) its general hyperplane section is a cDV singularity in the case of $d > 3$.

In the definition of cDV singularities, we assume the generality of hyperplane sections, but it is not necessary if one assumes that the point is originally singular. The following is well known, but we present a proof here for readers' convenience.

Lemma 3.3. *Assume that (X, x_0) is a germ of a d-dimensional hypersurface singularity of multiplicity 2 with $d \geq 3$, then (X, x_0) is a compound Du Val singularity if and only if there exist $(d-2)$ hyperplanes $H_1, \ldots, H_{d-2}$ such that $(X \cap H_1 \cap \cdots \cap H_{d-2}, x_0)$ is a Du Val singularity.*

Proof: The "only if" part is trivial and we will show the "if" part. Because the statement is local, we may assume that $(X, x_0) \subset (W, \underline{0})$, where $(W, \underline{0})$ is an open neighborhood of $x_0 = \underline{0}$ in $(\mathbb{A}_k^N, \underline{0})$, $(N = d + 1)$. Let $H_i = H_{\mathbf{a}_i} := \{(x_1, \ldots, x_N) \in W \mid \sum_{j=1}^N a_i^j x_j = 0\}$, where $\mathbf{a}_i = (a_i^1, \ldots, a_i^N) \in \mathbb{A}_k^N$. Define the family $\mathcal{S}$

$$\begin{array}{ccc} \mathcal{S} & \subset & X \times \mathbb{A}_k^{(d-2)N} \\ & \rho \searrow & \downarrow \tilde{\rho} \\ & & \mathbb{A}_k^{(d-2)N} \end{array}$$

so that for $\underline{\lambda} = (\underline{\lambda}_1, \ldots, \underline{\lambda}_{d-2}) \in \mathbb{A}_k^{(d-2)N}$ ($\underline{\lambda}_i = (\lambda_i^1, \ldots, \lambda_i^N) \in \mathbb{A}_k^N$),

$$\mathcal{S}_{\underline{\lambda}} := \rho^{-1}(\underline{\lambda}) = X \cap \left\{ \sum_{j=1}^N \lambda_i^j x_j = 0 \text{ for } i = 1, \ldots, d-2 \right\}.$$

The space $\mathcal{S}$ is a successive $(d-2)$ hyperplane cut of $X \times \mathbb{A}_k^{(d-2)N}$ around $(\underline{0}, \mathbf{a})$ ($\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_{d-2})$) and the $\mathrm{codim}_{\underline{0}}(\mathcal{S}_{\mathbf{a}}, X) = d - 2$, because $\mathcal{S}_{\mathbf{a}}$ is a surface with a Du Val singularity at $\underline{0}$. Therefore, $\mathrm{codim}_{(\underline{0}, \mathbf{a})}(\mathcal{S}, X \times \mathbb{A}^{(d-2)N}) = d - 2$. Then, ρ is flat around $(\underline{0}, \mathbf{a}) \in \mathcal{S}$ by a general theory (see for example, [M] p. 177, Corollary). By replacing W, therefore also X, by a sufficiently small neighborhood, there is an open neighborhood $U \subset \mathbb{A}_k^{(d-2)N}$ of $\mathbf{a}$ such that $\mathcal{S}|_U \to U$ is flat. Then, the family $\mathcal{S}|_U \to U$ is a flat family of surface singularities. As (X, x_0) is singular, $(\mathcal{S}_{\underline{\lambda}}, \underline{0})$ is also singular for every point $\underline{\lambda} \in U$. As is well known, Du Val singularities deform only to Du Val singularities (for example, see [B] or [KS]), and there is a neighborhood $U_0 \subset U$ of $\mathbf{a}$ such that the singularity $(\mathcal{S}_{\underline{\lambda}}, \underline{0})$ is a Du Val singularity for $\underline{\lambda} \in U_0$. This proves that the general $(d-2)$ hyperplane cut of (X, x_0) is Du Val. $\square$

Corollary 3.4. *Let (X, x_0) be a germ of a d-dimensional variety and $\widehat{R}$ be the M-adic completion of $R = \mathcal{O}_{X,x_0}$. Then, (X, x_0) is a compound Du Val singularity if and only if there exists $g_1, \ldots, g_{d-2} \in \widehat{R}$ such that $\mathrm{Spec}\widehat{R}/(g_1, \ldots, g_{d-2})$ is a Du Val singularity.*

Proof: The necessary condition is trivial. For the sufficient condition, let $(X, x_0) \subset (\mathbb{A}_k^{d+1}, \underline{0})$ and let $x_1, \ldots, x_{d+1}$ be a system of coordinates in $\mathbb{A}_k^{d+1}$. Note that the condition of (X, x_0) being a cDV singularity does not depend on the choice of the system of coordinates in $\mathbb{A}_k^{d+1}$ (this follows from a similar argument to that in the proof of Lemma 3.3). Let $g_1, \ldots, g_{d-2} \in \widehat{R}$ be such that $R' := \widehat{R}/(g_1, \ldots, g_{d-2})$ has a Du Val singularity. Then $\{g_1, \ldots, g_{d-2}\}$ is a regular sequence consisting of elements of multiplicity 1; hence, for fixed $n \gg 0$, after a change of affine coordinates in $\mathbb{A}_k^{d+1}$, we may suppose that $g_i = x_i \mod M^n$, for $1 \le i \le d-2$, where M is the maximal ideal of $\widehat{R}$. From this it follows that $\widehat{R}/(x_1, \ldots, x_{d-2})$ and $R/(x_1, \ldots, x_{d-2})$ have a Du Val singularity. Thus, (X, x_0) is a cDV singularity by Lemma 3.3. □

Lemma 3.5. *Let X be a d-dimensional variety and let $X' \subset X$ be a $(d-c)$-dimensional subvariety that is defined as the zero locus of c elements of $\mathcal{O}_X$. Let x_0 be a closed point in X'. If (X', x_0) is a top singularity, then (X, x_0) is also a top singularity.*

Moreover, given a germ of a d-dimensional variety (X, x_0), let $\widehat{R}$ be the M-adic completion of $R = \mathcal{O}_{X,x_0}$ (see notation in 2.3). Suppose that there exist $g_1, \ldots, g_c \in \widehat{R}$ such that the ring $R' = \widehat{R}/(g_1, \ldots, g_c)$ has dimension $d-c$ and

$$\inf_m \{ \ (m+1)(d-c) - \dim(\mathit{Spec}\ R')^0_m \ \} = d-c-1.$$

Then (X, x_0) is a top singularity.

Proof: By (4) and (13), it is sufficient to prove the second assertion. We may suppose that $\widehat{R} = k[[x_1, \ldots, x_N]]/I$ for an ideal $I \subset k[[x_1, \ldots, x_N]]$. Let $g_1, \ldots, g_c \in k[[x_1, \ldots, x_N]]$ be as in the lemma, and set $X' = \mathrm{Spec} R'$, where $R' = k[[x_1, \ldots, x_N]]/(I + (g_1, \ldots, g_c))$. Under the notation in 2.2, we obtain

$$\mathcal{O}_{(X')^0_m} = \mathcal{O}_{(X)^0_m} \ / \ (\{(G_i)^0_1, \ldots, (G_i)^0_m\}_{i=1}^c),$$

where we identify $(G_i)^0_n \in k[\underline{X}_1, \ldots, \underline{X}_m]$ with its class in $\mathcal{O}_{(X)^0_m}$. Since $\mathcal{O}_{X^0_m}$ is a catenary ring, applying Krull's theorem, we obtain

$$\dim \mathcal{O}_{X^0_m} = \dim \mathcal{O}_{(X')^0_m} + \mathrm{ht}\ (\{(G_i)^0_1, \ldots, (G_i)^0_m\}_{i=1}^c) \le \dim \mathcal{O}_{(X')^0_m} + mc. \tag{14}$$

Therefore, from (13) we have

$$\widehat{\mathrm{mld}}(x_0; X, \mathcal{J}_X) = \inf_m \{(m+1)d - \dim(\mathrm{Spec}\widehat{R})^0_m\}$$

$$\ge \inf_m \{(m+1)(d-c) - \dim(\mathrm{Spec}R')^0_m\} + c = (d-c-1) + c = d-1.$$

In addition, X has a singularity at x_0, because (X', x_0) is singular, hence $\widehat{\mathrm{mld}}(x_0; X, \mathcal{J}_X) = d-1$ by Proposition 2.10. □

Lemma 3.6. *If X has a top singularity at x_0, then X is locally a hypersurface of multiplicity 2 at x_0.*

Proof: From (13) it follows that a germ of an algebraic variety (X, x_0) is a top singularity if and only if

$$\dim X^0_m \le md + 1 \qquad \text{for every } m \ge 1 \tag{15}$$

and the equality holds at least for an integer m. Suppose that (X, x_0) is a top singularity. We may suppose that X is affine. Let N be the embedding dimension of X at x_0, then $N \ge d+1$ because x_0 is a singular point of X. Besides, with the notation in 2.2, we have $X^0_1 = \mathrm{Spec}\ k[\underline{X}_1]$; hence, $\dim X^0_1 = N$. Thus, inequality

(15) for $m = 1$ implies that $N = d + 1$, *i.e.,* X is locally a hypersurface at x_0. Let X be defined by $f(x_1, \ldots, x_{d+1}) = 0$, then,

$$X_2^0 = \text{Spec } k[\underline{X}_1, \underline{X}_2] \;/\; (F_2^0),$$

and we have $\dim X_2^0 = 2d + 1$ (resp. $\dim X_2^0 = 2(d + 1)$) if $F_2^0 \neq 0$ (resp. $F_2^0 = 0$). Therefore, inequality (15) for $m = 2$ implies that $F_2^0 \neq 0$, *i.e.*, $\text{mult}_{x_0} f = 2$. □

Corollary 3.7. *A singularity (X, x_0) is a top singularity if and only if*

$$\dim X_m^0 = md + 1 \qquad \text{for every } m \geq 1. \tag{16}$$

Proof: The sufficient condition is clear. For the necessary one, suppose that (X, x_0) is a top singularity. Then, X is locally a hypersuperface of multiplicity 2 at x_0, and let it be defined by $f(x_1, \ldots, x_{d+1}) = 0$. Then, $\dim X_1^0 = d + 1$ and for $m \geq 2$, we have

$$\mathcal{O}_{X_m^0} = \mathcal{O}_{X_{m-1}^0}[\underline{X}_m] \;/\; (F_m^0),$$

and hence, $\dim X_m^0 = \dim X_{m-1}^0 + d + \delta_m$, where $\delta_m = 0$ (resp. $\delta_m = 1$) if F_m^0 is not a zero divisor (resp. F_m^0 is a zero divisor) in $\mathcal{O}_{X_{m-1}^0}[\underline{X}_m]$. Therefore, $\dim X_m^0 = md + 1 + \sum_{r=2}^m \delta_r$. But (X, x_0), a top singularity, implies that (15) holds, and hence $\sum_{r=2}^m \delta_r \leq 0$. Thus, $\sum_{r=2}^m \delta_r = 0$ and (16) holds. □

Definition 3.8. Let X be the hypersurface defined by $x_1 x_2 = 0$ in $\mathbb{A}^{d+1} (d \geq 1)$, where $\{x_1, x_2\}$ is a part of the coordinate system of $\mathbb{A}^{d+1}$. Then, the singularity $(X, 0)$ is called a *normal crossing double* singularity (sometimes we call it an ncd singularity).

Let X be the hypersurface defined by $x_1^2 - x_2^2 x_3 = 0$ in $\mathbb{A}^{d+1} (d \geq 1)$. Then, the singularity $(X, 0)$ is called a *pinch point*.

Example 3.9. Next we give an example of a top singularity of dimension $d = 1$: Let X be a plane curve with an ordinary node, *i.e.,* locally it is defined by $x_1 x_2 = 0$ in $\mathbb{A}_k^2$, and let us consider its germ $(X, \underline{0})$ at $\underline{0}$. Then, by inversion of adjunction, we have

$$\widehat{\text{mld}}(\underline{0}; X, \mathcal{J}_X) = \text{mld}(\underline{0}; \mathbb{A}^2, I_X) = \text{mld}(\underline{0}; \mathbb{A}^2, X).$$

It is well known that the right hand side is 0, *i.e.,* $(X, \underline{0})$ is a top singularity.

We give here another proof by using jet schemes. For $m \geq 0$, we have

$$X_m^0 = \text{Spec } k[\underline{X}_1, \ldots, \underline{X}_m] \;/\; (\{\sum_{1 \leq i \leq n-1} X_{1,i} X_{2,n-i}\}_{1 \leq n \leq m}).$$

It follows that X_m^0 has m irreducible components given by

$$X_{1,1} = X_{1,2} = \ldots = X_{1,r_1} = X_{2,1} = \ldots = X_{2,r_2} = 0 \quad \text{for } r_1, r_2 \geq 0,\ r_1 + r_2 = m - 1.$$

Thus, each irreducible component has dimension $2m - (m - 1) = m + 1$, and hence, $\dim X_m^0 = m + 1$ for $m \geq 0$. Therefore, $(X, \underline{0})$ is a top singularity.

Example 3.10. We present a two-dimensional example in the following: Let $X \subset \mathbb{A}^3$ be the hypersurface defined by $x_1^2 - x_2^2 x_3 = 0$. Then, $(X, \underline{0})$ is a top singularity. Indeed, let $\varphi : A' \to A = \mathbb{A}^3$ be the blow-up at the singular locus of X and let E be the exceptional divisor for φ, then the strict transform Y of X in A' is non-singular and crosses E normally. By the inversion of adjunction, we have

$$\widehat{\text{mld}}(\underline{0}; X, \mathcal{J}_X) = \text{mld}(\underline{0}; A, I_X).$$

Here as A' is also non-singular, the right hand side is

$$\text{mld}(\underline{0}; A, I_X) = \text{mld}(\underline{0}; A, X) = \text{mld}(\varphi^{-1}(\underline{0}); A', Y + E),$$

because $I_X \mathcal{O}_{A'} = \mathcal{O}'_A(-Y-2E)$, $K_{A'/A} = E$ and $K_{\overline{A}/A} = K_{\overline{A}/A'} + f^*(K_{A'/A})$, where $f : \overline{A} \to A'$ is a resolution. As $\varphi^{-1}(\underline{0})$ is a curve that does not contain the double locus of $Y+E$, it is well known that

$$\mathrm{mld}(\varphi^{-1}(\underline{0}); A', Y+E) = 1.$$

Proposition 3.11. *A normal crossing double singularity and a pinch point are top singularities.*

Proof: From Example 3.9. for $d=1$, an ncd singularity $(X,0) = (C,0)$ is a top singularity. Let $d \geq 2$ and $(X,0)$ be a d-dimensional ncd singularity. Then, the $(d-1)$ successive general hyperplane cut gives an ordinary double point $(C,0)$ of a curve C. Then, by Lemma 3.5, $(X,0)$ is a top singularity.

For the pinch point of dimension d, by the same argument as above, we can reduce the discussion to the two-dimensional case (Example 3.10). □

Remark 3.12. For $d \geq 2$, recall that if $X \subset \mathbb{A}_k^{d+1}$ is a normal hypersurface and x_0 is a closed point of X, then

$$\begin{aligned}\widehat{\mathrm{mld}}(x_0; X, \mathcal{J}_X) = \mathrm{mld}(x_0; X, \mathcal{O}_X) = \\ = \inf\{k_E + 1 \mid \ \nu_E \text{ divisorial valuation centered at } x_0\}.\end{aligned}$$

Note that if $\pi : Y \to X$ and $\pi' : Y' \to X$ are two desingularizations of X and Y' dominates Y, let $\rho : Y' \to Y$ be such that $\pi' = \rho \circ \pi$, then we have $K_{Y'/X} = K_{Y'/Y} + \rho^*(K_{Y/X})$ and $K_{Y'/Y}$ is effective. Therefore, given a normal hypersurface $X \subset \mathbb{A}_k^{d+1}$ of dimension $d \geq 2$, to prove that (X, x_0) is a top singularity, it suffices to show that there exists a desingularization $\pi : Y \to X$ such that

$$\inf\{k_E + 1 \mid \ E \text{ prime divisor on } Y \text{ such that } \pi(E) = x_0\} = d-1. \tag{17}$$

Example 3.13. Equality (17) is satisfied for the minimal desingularizations of all rational double points of dimension 2 (also called Du Val singularities) because they are canonical singularities of dimension $d=2$. The following is a list of rational double points, for each of them, the completion $\widehat{\mathcal{O}_{X,\underline{0}}}$ of the local ring $\mathcal{O}_{X,\underline{0}}$ of its germ at $\underline{0}$ is described as a quotient of the ring of series $k[[x_1, x_2, x_3]]$. More precisely, for each of the types of the rational double points on the left hand side, there exist $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ generating the maximal ideal of $\widehat{\mathcal{O}_{X,\underline{0}}}$ and satisfying the equation on the right hand side (recall that char $k=0$):

$$\begin{array}{ll}\mathbf{A}_n (n \geq 1): & \mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^{n+1} = 0 \\ \mathbf{D}_n (n \geq 4): & \mathbf{x}_1^2 + \mathbf{x}_2^2\mathbf{x}_3 + \mathbf{x}_3^{n-1} = 0 \\ \mathbf{E}_6 : & \mathbf{x}_1^2 + \mathbf{x}_2^3 + \mathbf{x}_3^4 = 0 \\ \mathbf{E}_7 : & \mathbf{x}_1^2 + \mathbf{x}_2^3 + \mathbf{x}_2\mathbf{x}_3^3 = 0 \\ \mathbf{E}_8 : & \mathbf{x}_1^2 + \mathbf{x}_2^3 + \mathbf{x}_3^5 = 0\end{array}$$

Proposition 3.14. *A compound Du Val singularity is a top singularity.*

Proof: Let (X, x_0) be a compound Du Val singularity of dimension $d \geq 3$. Then, a successive $(d-2)$ hyperplane cut produces a Du Val singularity. As in the previous example, Du Val singularities are top singularities. By Lemma 3.5, we obtain that (X, x_0) is a top singularity. □

3.15. We will prove that cDV, ncd, and pinch points are all top singularities. Recall that given $f(x_1, \ldots, x_{d+1}) \in k[x_1, \ldots, x_{d+1}]$ (resp. $f \in k[[x_1, \ldots, x_{d+1}]]$), if $\mathrm{in} f$ denotes the initial form of f in the graded ring $k[x_1, \ldots, x_{d+1}]$ (resp. $k[[x_1, \ldots, x_{d+1}]]$), with the usual graduation, then the smallest possible dimension τ of a linear subspace V_0 of $V = kx_1 + \ldots + kx_{d+1}$ such that $\mathrm{in} f$ lies in the subalgebra $k[V_0]$ of

$k[x_1,\dots,x_{d+1}]$ is an invariant of the germ $(X,\underline{0})$ of the hypersurface $X\subset\mathbb{A}_k^{d+1}$ at $\underline{0}$ defined by $f(x_1,\dots,x_{d+1})=0$ (resp. of Spec $k[[x_1,\dots,x_{d+1}]]/(f)$) ([Hi1], chap. III). We denote it by $\tau(X,\underline{0})$ (resp. by $\tau(f)$). Given a germ (X,x_0) of a hypersurface in $\mathbb{A}_k^{d+1}$ at a closed point x_0, the τ-invariant $\tau(X,x_0)$ is defined as the τ-invariant of the germ of a hypersurface obtained after a translation of x_0 to $\underline{0}$.

Lemma 3.16. *Let $(X,\underline{0})$ be the germ of a hypersurface $X\subset\mathbb{A}_k^{d+1}$ of multiplicity* 2 *at $\underline{0}$. Then, there exist $\mathbf{x}_1,\dots,\mathbf{x}_{d+1}\in\widehat{\mathcal{O}_{X,\underline{0}}}$ generating its maximal ideal and satisfying*

$$\mathbf{x}_1^2+\ldots+\mathbf{x}_\tau^2+g(\mathbf{x}_{\tau+1},\dots,\mathbf{x}_{d+1})=0$$

with $\tau=\tau(X,\underline{0})$, $g(x_{\tau+1},\dots,x_{d+1})\in k[[x_{\tau+1},\dots,x_{d+1}]]$ and either $g=0$ or mult $g\geq 3$.

Proof: This is well known (for example, see [KM] 4.24 and 4.25). Actually, iterating the procedures Steps $1-3$ in 4.25 of [KM], we obtain the required equation. $\square$

Proposition 3.17. *Let (X,x_0) be a germ of a hypersurface of multiplicity* 2 *and $\tau(X,x_0)>1$. Then, (X,x_0) is either an ncd singularity or a cDV singularity; therefore, it is a top singularity.*

Proof: As in Lemma 3.16, let $\mathbf{x}_1,\dots,\mathbf{x}_{d+1}\in\widehat{\mathcal{O}_{X,\underline{0}}}$ generate its maximal ideal and satisfy

$$\mathbf{x}_1^2+\ldots+\mathbf{x}_\tau^2+g(\mathbf{x}_{\tau+1},\dots,\mathbf{x}_{d+1})=0$$

with $\tau=\tau(X,\underline{0})$, $g(x_{\tau+1},\dots,x_{d+1})\in k[[x_{\tau+1},\dots,x_{d+1}]]$ and either $g=0$ or mult $g\geq 3$.

If $\tau\geq 3$, let $R':=\widehat{R}/(\mathbf{x}_4,\dots,\mathbf{x}_{d+1})$, where $\mathbf{x}_i$ denotes the class of x_i in $\widehat{R}=\widehat{\mathcal{O}_{X,\underline{0}}}$. Then, Spec$R'$ is defined in Spec$k[[x_1,x_2,x_3]]$ by

$$x_1^2\ +\ x_2^2\ +\ x_3^2=0.$$

Hence, it has an $\mathbf{A}_1$-singularity at $\underline{0}$, and therefore, (X,x_0) is a cDV singularity (Corollary 3.4).

If $\tau=2$ and $g=0$, then (X,x_0) is an ncd singularity.

If $\tau=2$ and $g\neq 0$, then there exists $\underline{\lambda}=(\lambda_4,\dots,\lambda_{d+1})\in\mathbb{A}_k^{d-1}$ such that $g(x_3,\lambda_4x_3,\dots,\lambda_{d+1}x_3)$ is non-zero and its multiplicity is $m=\mathrm{mult}_{\underline{0}}\,g(x_3,\dots,x_{d+1})$. Let $R':=\widehat{R}/(\mathbf{x}_4-\lambda_4\mathbf{x}_3,\dots,\mathbf{x}_{d+1}-\lambda_{d+1}\mathbf{x}_3)$, then Spec$R'$ is defined in Spec$k[[x_1,x_2,x_3]]$ by

$$x_1^2\ +\ x_2^2\ +\ u\ x_3^m=0,$$

where u is a unit in $k[[x_3]]$; hence, SpecR' has an $\mathbf{A}_{m-1}$-singularity (see Example 3.13); thus, (X,x_0) is a cDV singularity. $\square$

3.18. Let $(X,\underline{0})$ be a germ of a hypersurface $X\subseteq\mathbb{A}_k^{d+1}$ of multiplicity 2 and $\tau(X,\underline{0})=1$. Let $\mathbf{x}_1,\dots,\mathbf{x}_{d+1}$ be generating the maximal ideal of $\widehat{\mathcal{O}_{X,\underline{0}}}$ and satisfying

$$\mathbf{x}_1^2+g(\mathbf{x}_2,\dots,\mathbf{x}_{d+1})=0, \tag{18}$$

where $g(x_2,\dots,x_{d+1})\in k[[x_2,\dots,x_{d+1}]]$ and since X is reduced, $g\neq 0$ and mult $g\geq 3$ (Lemma 3.16). Let us consider the germ of the hypersurface $g(x_2,\dots,x_{d+1})=0$ at $\underline{0}$ in Spec $k[[x_1,\dots,x_{d+1}]]$. Although this germ depends on the choice of $x_1,\dots,x_{d+1}$, its multiplicity $m_2:=$ mult g, and its τ-invariant at $\underline{0}$, let it be τ_2, which only depends on $(X,\underline{0})$ (this follows from [Hi2]. See Remark 3.19). Given a germ (X,x_0) of a hypersurface in $\mathbb{A}_k^{d+1}$ at a closed point x_0, we define $m_2(X,x_0)$ and $\tau_2(X,x_0)$ to be the invariants defined as before, after a translation of x_0 to $\underline{0}$.

Remark 3.19. In [Hi2], the following combinatorial object has been considered: Given $f \in k[[x_1,\dots,x_{d+1}]]$, let $f = \sum_{n_1,\underline{n}} c_{n_1,\underline{n}}\ x_1^{n_1}\underline{x}^{\underline{n}}$, where $c_{n_1,\underline{n}} \in k$, $\underline{x} = (x_2,\dots,x_{d+1})$ and $(n_1,\underline{n})$ runs in $\mathbb{Z}_{\geq 0}\times(\mathbb{Z}_{\geq 0})^d$, let $\mathrm{Supp}(f;\underline{x};x_1) = \{(n_1,\underline{n})$ such that $c_{n_1,\underline{n}} \neq 0\}$, and let $p_m : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ be the projection from $(m,\underline{0})$, where $m := \mathrm{mult} f$. Then

$$p_m(\mathrm{Supp}(f;\underline{x};x_1)) \;=\; \left\{\frac{1}{m-n_1}\,\underline{n}\ /\ c_{n_1,\underline{n}} \neq 0\right\}.$$

Let $\Delta(f;\underline{x};x_1)$ be the convex hull of $p_m(\mathrm{Supp}(f;\underline{x};x_1)) + (\mathbb{R}_{\geq 0})^d$, and let

$$\Delta(f;\underline{x}) := \cap_{x'}\Delta(f;\underline{x};x'),$$

where x' runs in the set of elements of $k[[x_1,\dots,x_{d+1}]]$ such that $x', x_2,\dots x_{d+1}$ is a regular system of parameters of $k[[x_1,\dots,x_{d+1}]]$. The combinatorial object $\Delta(f;\underline{x})$ is called the *(first) characteristic polyhedron of f with respect to $\underline{x}$* ([Hi2], def. 1.2).

Now, let $f = x_1^2 + g(x_2,\dots,x_{d+1})$, where $g \in k[[x_2,\dots,x_{d+1}]]$, $g \neq 0$, and mult $g \geq 3$ (see (20)). Because there is no term in x_1 in the previous expression of f, we have $\Delta(f;\underline{x};x_1) = \Delta(f;\underline{x})$ ([Hi2] Theorem 4.8). Suppose that $x_1',\dots,x_{d+1}'$ is another regular system of parameters of $k[[x_1,\dots,x_{d+1}]]$ such that

$$x_1^2 + g(x_2,\dots,x_{d+1}) = u\ ((x_1')^2 + g(x_2',\dots,x_{d+1}')), \tag{19}$$

where u is a unit in $k[[x_1,\dots,x_{d+1}]]$ and $g' \in k[[x_2',\dots,x_{d+1}']]$, mult $g' \geq 3$. Then, considering the initial forms in the graded ring $k[[x_1,\dots,x_{d+1}]]$, with the usual graduation, it follows that $x_1' = v(x_1+h)$, where v is a unit and $h \in (x_1,\dots,x_{d+1})^2$. On the other hand, by the definition of $\Delta(f;\underline{x};x_1)$ one can naturally consider the graduation on $k[[x_1,\dots,x_{d+1}]]$ defined by the monomial valuation on $k[[x_1,\dots,x_{d+1}]]$ given by $\nu(x_1) = \frac{1}{2}$, $\nu(x_i) = \frac{1}{\mathrm{mult}\ g}$ for $2 \leq i \leq d+1$. Then, considering the initial form with respect to this graduation on both sides of (19), we obtain that $\nu(h) > \frac{1}{2}$, $\nu(x_1') = \nu(x_1)$, and $\mathrm{in}_\nu g = \mathrm{in}_\nu g'$, which implies mult $g =$ mult g' and $\tau(g) = \tau(g')$.

Lemma 3.20. *Let $(X,\underline{0})$ be a germ of a hypersurface in $\mathbb{A}_k^{d+1}$ of multiplicity 2 and $\tau(X,\underline{0}) = 1$. If $(X,\underline{0})$ is a top singularity then $d \geq 2$ and $m_2(X,\underline{0}) = 3$.*

Proof: Let $\mathbf{x}_1,\dots,\mathbf{x}_{d+1}$ be generating the maximal ideal of $\widehat{\mathcal{O}_{X,\underline{0}}}$ and $g \in k[[x_2,\dots,x_{d+1}]]$ such that (18) holds. Then, we have

$$X_1^0 \cong \mathrm{Spec}\ k[\underline{X}_1] \qquad X_2^0 \cong \mathrm{Spec}\ k[\underline{X}_1,\underline{X}_2]\ /\ ((X_{1,1})^2)$$
$$X_3^0 \cong \mathrm{Spec}\ k[\underline{X}_1,\underline{X}_2,\underline{X}_3]\ /\ ((X_{1,1})^2, G_3^0)$$

(see (3) and (4)). Moreover, $X_{1,1}$ does not divide G_3^0; hence, $\dim X_3^0 = 3d+1$ (resp. $\dim X_3^0 = 3d+2$) if $G_3^0 \neq 0$ (resp. $G_3^0 = 0$). Therefore, if $(X,\underline{0})$ is a top singularity then $G_3^0 \neq 0$ (see (16)) or equivalently, $m_2(X,\underline{0}) = \mathrm{mult}_{\underline{0}} g = 3$.

Now let us show that if $(X,\underline{0})$ is a top singularity, then $d \geq 2$. In fact, suppose that $d = 1$ and $m_2(X,\underline{0}) = 3$. Then, there exist $\mathbf{x}_1,\mathbf{x}_2$ generating the maximal ideal of $\widehat{\mathcal{O}_{X,\underline{0}}}$ such that $\mathbf{x}_1^2 + \mathbf{x}_2^3 = 0$. In fact, there exist $\mathbf{x}_1,\mathbf{x}_2 \in \widehat{\mathcal{O}_{X,\underline{0}}}$ such that $\mathbf{x}_1^2 + g(\mathbf{x}_2) = 0$, where $g(x_2) \in k[[x_2]]$ has multiplicity 3; hence, $g(x_2) = x_2^3\ u(x_2)$, where $u(x_2)$ is a unit in $k[[x_2]]$. Because k is algebraically closed, there exists $v(x_2) \in k[[x_2]]$ such that $v(x_2)^3 = u(x_2)$. Then, replacing $\mathbf{x}_2$ by $v(\mathbf{x}_2)\ \mathbf{x}_2$, we have $\mathbf{x}_1^2 + \mathbf{x}_2^3 = 0$. Therefore,

$$(X_4^0)_{\mathrm{red}} \cong \mathrm{Spec}\ k[\underline{X}_1,\dots,\underline{X}_4]\ /\ (X_{1,1}, X_{2,1}, X_{1,2})$$
$$(X_5^0)_{\mathrm{red}} \cong \mathrm{Spec}\ k[\underline{X}_1,\dots,\underline{X}_5]\ /\ (X_{1,1}, X_{2,1}, X_{1,2}).$$

Hence, $\dim X_5^0 = 7 > 5+1$ and $(X,\underline{0})$ is not a top singularity. This concludes the proof. $\square$

Proposition 3.21. *Let (X, x_0) be a germ of a hypersurface in $\mathbb{A}_k^{d+1}$ of multiplicity 2 and $\tau(X, x_0) = 1$. If $m_2(X, x_0) = 3$ and $\tau_2(X, x_0) > 1$, then (X, x_0) is either a cDV singularity or a pinch point; therefore, it is a top singularity.*

Proof: We may suppose that the point x_0 is the origin $\underline{0} \in \mathbb{A}_k^{d+1}$. We can take generators $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$ of the maximal ideal of $\widehat{\mathcal{O}_{X,\underline{0}}}$ such that

$$\mathbf{x}_1^2 + g_3(\mathbf{x}_2, \ldots, \mathbf{x}_{\tau_2+1}) + h(\mathbf{x}_2 \ldots, \mathbf{x}_{d+1}) = 0.$$

Here $g_3(x_2, \ldots, x_{\tau_2+1}) \in k[x_2, \ldots, x_{\tau_2+1}]$ is homogeneous of degree 3 and $h \in k[[x_2, \ldots, x_{d+1}]]$ has multiplicity ≥ 4. Let $R' = \widehat{R}/(\mathbf{x}_{\tau_2+2}, \ldots, \mathbf{x}_{d+1})$, then Spec$R'$ is defined in Spec$k[[x_1, \ldots, x_{\tau_2+1}]]$ by

$$x_1^2 + g_3(x_2, \ldots, x_{\tau_2+1}) + g_4(x_2 \ldots, x_{\tau_2+1}) = 0, \tag{20}$$

where g_3 is homogeneous of degree 3 as stated before and mult$g_4 \geq 4$. Next, take a general combination $\lambda = (\lambda_2, \ldots, \lambda_{\tau_2+1}) \in \mathbb{A}_k^{\tau_2}$ so that the hyperplane H in $\mathbb{P}_k^{\tau_2-1}$ defined by $\sum_{i=2}^{\tau_2+1} \lambda_i x_i = 0$ does not contain any irreducible components of the hypersurface G defined by $g_3 = 0$. Here we note that the hypersurface G is not a triple plane because $\tau_2 > 1$ and by the definition of τ_2. Therefore, we may also assume that $G \cap H$ is not a triple hyperplane in H. Now let $R'' = R'/(\sum_{i=2}^{\tau_2+1} \lambda_i \mathbf{x}_i)$. As λ is general, we may assume that $\lambda_{\tau_2+1} \neq 0$, and therefore, we can eliminate x_{τ_2+1}, *i.e.,* SpecR'' is defined in Spec$k[[x_1, \ldots, x_{\tau_2}]]$ by

$$x_1^2 + g_3'(x_2, \ldots, x_{\tau_2}) + g_4'(x_2 \ldots, x_{\tau_2}) = 0,$$

where g_3' and g_4' have the same properties as g_3 and g_4 in (20), respectively. By reiterating this argument, we obtain a two-dimensional singularity whose germ is

$$\widehat{R} \;\cong\; k[[x_1, x_2, x_3]] \;/\; (\; x_1^2 \;+\; g(x_2, x_3)\;), \tag{21}$$

where $g(x_2, x_3) \in k[[x_2, x_3]]$, mult $g(x_2, x_3) = 3$, and by the above argument, in$g(x_2, x_3)$ does not give a triple point in $\mathbb{P}_k^1$, *i.e.,* in$g(x_2, x_3)$ has at least two different linear factors. In this case, after a possible change of the regular system of parameters of $k[[x_2, x_3]]$, we may suppose that $g(x_2, x_3) = x_3(x_2^2 + v x_3^m)$, where $m \geq 2$ and $v \in k[[x_3]]$ is either a unit or 0 ([KM], Step 4 in 4.25). Here if v is a unit, then Spec$\widehat{R}$ has $\mathbf{D}_{m+2}$-singularity and if $v = 0$, then Spec$\widehat{R}$ has a pinch point. $\square$

3.22. Let $(X, \underline{0})$ be a germ of a hypersurface $X \subseteq \mathbb{A}_k^{d+1}$ of multiplicity 2, $\tau(X, \underline{0}) = 1$, $m_2(X, \underline{0}) = 3$, and $\tau_2(X, \underline{0}) = 1$. Then, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$ generating the maximal ideal of $\widehat{\mathcal{O}_{X,\underline{0}}}$ such that

$$\mathbf{x}_1^2 \;+\; \mathbf{x}_2^3 \;+\; g_3(\mathbf{x}_3, \ldots, \mathbf{x}_{d+1})\, \mathbf{x}_2 \;+\; g_4(\mathbf{x}_3, \ldots, \mathbf{x}_{d+1}) = 0, \tag{22}$$

where $g_i \in k[[x_3, \ldots, x_{d+1}]]$ and mult$_{\underline{0}}$ $g_i \geq i$, for $i = 3, 4$. In fact, there exist $x_1, \ldots, x_{d+1}$ whose classes in $\widehat{\mathcal{O}_{X,\underline{0}}}$ generate the maximal ideal, and $g \in k[[x_2, \ldots, x_{d+1}]]$ such that (18) holds. Moreover, because mult $g = m_2(X, x_0) = 3$ and $\tau_2(X, x_0) = 1$, by Weierstrass' preparation theorem and after a Tschirnhausen transformation, we may suppose that

$$g(x_2, \ldots, x_{d+1}) = u\;\left(x_2^3 \;+\; g_3(x_3, \ldots, x_{d+1})\; x_2 \;+\; g_4(x_3, \ldots, x_{d+1})\right),$$

where u is a unit in $k[[x_2, \ldots, x_{d+1}]]$ and $g_i \in k[[x_3, \ldots, x_{d+1}]]$ is such that mult $g_i \geq i$ for $i = 3, 4$. Replacing x_1 by $v x_1$, where v is a unit in $k[[x_2, \ldots, x_{d+1}]]$ such that $v^2 = u$, and considering the equality induced on the classes $\mathbf{x}_i$ of x_i in $\widehat{\mathcal{O}_{X,\underline{0}}}$, we obtain (22).

Given $g_3, g_4 \in k[[x_3, \dots, x_{d+1}]]$ as in (22), let

$$m_3(g_3, g_4) := 6 \ \min \left\{ \frac{\text{mult } g_3}{2}, \ \frac{\text{mult } g_4}{3} \right\}.$$

Note that $m_3(g_3, g_4) \in \mathbb{N} \cup \{\infty\}$ and $\frac{1}{6}\ m_3(g_3, g_4) > 1$. Moreover, $m_3(g_3, g_4)$ is an invariant of $(X, \underline{0})$. This follows from [Hi2] (see Remark 3.19). Let $m_3(X, \underline{0})$ denote this invariant.

Proposition 3.23. *Let (X, x_0) be a germ of a hypersurface in $\mathbb{A}_k^{d+1}$ of multiplicity 2 and $\tau(X, x_0) = 1$, $m_2(X, x_0) = 3$, and $\tau_2(X, x_0) = 1$. Then, the following are equivalent:*

(i) *(X, x_0) is a top singularity,*

(ii) *there exist $\mathbf{x}_1, \dots, \mathbf{x}_{d+1}$ generating the maximal ideal of $\widehat{\mathcal{O}_{X,x_0}}$ such that*

$$\mathbf{x}_1^2 \ + \ \mathbf{x}_2^3 + g_3(\mathbf{x}_3, \dots, \mathbf{x}_{d+1}) \ \mathbf{x}_2 \ + \ g_4(\mathbf{x}_3, \dots, \mathbf{x}_{d+1}) \ = \ 0.$$

where $g_i \in k[[x_3, \dots, x_{d+1}]]$, mult $g_i \geq i$ *for $i = 3, 4$ and either* mult $g_3 = 3$ *or $4 \leq$* mult $g_4 \leq 5$,

(iii) $\frac{1}{6} m_3(X, x_0) < 2$,

(iv) *(X, x_0) is a cDV singularity.*

Proof: Implication (iv) $\Rightarrow$ (i) is obvious by Proposition 3.14. We will show (ii) $\Leftrightarrow$ (iii), and then (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iv).

Let $R = \mathcal{O}_{X,x_0}$, then we have

$$\widehat{R} \ \cong \ k[[x_1, \dots, x_{d+1}]] \ / \ (x_1^2 \ + \ x_2^3 \ + \ g_3 \ x_2 \ + \ g_4), \tag{23}$$

where $g_3, g_4 \in k[[x_3, \dots, x_{d+1}]]$ and $\mu_i :=$ mult $g_i \geq i$, for $i = 3, 4$ (see 3.22). Note that $\mu_i = \infty$ iff $g_i = 0$. Moreover, by the definition of $m_3(X, x_0)$ (see the paragraph before Proposition 3.23 and recall that $\frac{1}{6} m_3(X, x_0) > 1$), the condition $\frac{1}{6} m_3(X, x_0) < 2$ is equivalent to the assertion that either $\mu_3 = 3$ or $4 \leq \mu_4 \leq 5$. Thus, (ii) is equivalent to (iii).

To prove (i) $\Rightarrow$ (ii), let us argue by contradiction. Suppose that $\mu_3 \geq 4$ and $\mu_4 \geq 6$. Then, applying (3) and (4), we obtain

$$X_1^0 \cong \text{Spec } k[\underline{X}_1] \qquad (X_2^0)_{\text{red}} \cong \text{Spec } k[\underline{X}_1, \underline{X}_2] \ / \ (X_{1,1})$$
$$(X_3^0)_{\text{red}} \cong \text{Spec } k[\underline{X}_1, \underline{X}_2, \underline{X}_3] \ / \ (X_{1,1}, X_{2,1})$$
$$(X_4^0)_{\text{red}} \cong \text{Spec } k[\underline{X}_1, \dots, \underline{X}_4] \ / \ (X_{1,1}, X_{2,1}, X_{1,2})$$
$$(X_5^0)_{\text{red}} \cong \text{Spec } k[\underline{X}_1, \dots, \underline{X}_5] \ / \ (X_{1,1}, X_{2,1}, X_{1,2}).$$

That is, with the notation in 2.2, we have $F_5^0 \in (X_{1,1}, X_{2,1}, X_{1,2})$, where $f = x_1^2 + x_2^3 + g_3 \ x_2 + g_4$ (see (23)). Therefore, $\dim X_5^0 = 5d + 2$, and hence, (X, x_0) is not a top singularity (see (15)).

To prove (ii) $\Rightarrow$ (iv), let us suppose that either $\mu_3 = 3$ or $4 \leq \mu_4 \leq 5$. Let $\underline{\lambda} = (\lambda_4, \dots, \lambda_{d+1}) \in \mathbb{A}_k^{d-2}$ be such that

$$\text{mult } g_i(x_3, \lambda_4 x_3, \dots, \lambda_{d+1} x_3) = \text{mult } g_i(x_3, x_4, \dots, x_{d+1}) = \mu_i \quad \text{for } i = 3, 4.$$

Hence, for $i = 3, 4$, if $g_i \neq 0$ then

$$g_i(x_3, \lambda_4 x_3, \dots, \lambda_{d+1} x_3) = u_i \ x_3^{\mu_i}, \tag{24}$$

where u_i is a unit in $k[[x_3]]$. Let us consider

$$R' := \widehat{R} \ / \ (\mathbf{x}_4 - \lambda_4 \mathbf{x}_3, \ \dots \ , \mathbf{x}_4 - \lambda_{d+1} \mathbf{x}_{d+1}),$$

where $\mathbf{x}_i$ is the class of x_i in $\widehat{R}$, $3 \leq i \leq d+1$. By Lemma 3.3, it is sufficient to prove that SpecR' has a Du Val singularity.

Note that, by (23) and (24),

$$R' \;\cong\; k[[x_1,x_2,x_3]] \;/\; (x_1^2 \;+\; x_2^3 \;+\; g_3'(x_3)\;x_2 \;+\; g_4'(x_3)),$$

where, for $i = 3,4$, we have $g_i'(x_3) = 0$ if $\mu_i = \infty$ and $g_i'(x_3) = u_i\; x_3^{\mu_i}$ if $\mu_i < \infty$. Here g_i' has the same property on the multiplicity as g_i in (ii). Then, by Steps 5 – 8 of 4.25 in [KM], we obtain that SpecR' has $\mathbf{E}_n$-singularity ($n = 6,7,8$). $\square$

The following summarizes the discussions of characterization of a top singularity (Proposition 3.17, Lemma 3.20, Proposition 3.21, Proposition 3.23).

Corollary 3.24. *A germ of a variety (X,x_0) of dimension $d = 1$ is a top singularity if and only if it is the germ of a hypersurface in $\mathbb{A}_k^2$ of multiplicity 2 at $x_0$and $\tau(X,x_0) > 1$.*

A germ of a variety (X,x_0) of dimension $d \geq 2$ is a top singularity if and only if it is the germ of a hypersurface in $\mathbb{A}_k^{d+1}$ of multiplicity $m_1(X,x_0) = 2$ at x_0 such that one of the following holds:

(i) $\tau(X,x_0) > 1$

(ii) $\tau(X,x_0) = 1,\ \dfrac{m_2(X,x_0)}{m_1(X,x_0)!} < 2$ *and* $\tau_2(X,x_0) > 1$

(iii) $\tau(X,x_0) = 1,\ \dfrac{m_2(X,x_0)}{m_1(X,x_0)!} < 2,\ \tau_2(X,x_0) = 1$ *and* $\dfrac{m_3(X,x_0)}{m_2(X,x_0)!} < 2.$

Proof: For $d = 1$, the result follows from Proposition 3.11 and Lemmas 3.6 and 3.20. For $d \geq 2$, note that, if $\tau(X,x_0) = 1$, then $\frac{m_2(X,x_0)}{m_1(X,x_0)!} = \frac{1}{2}m_2(X,x_0)$ is well defined and is always > 1 (see 3.18 and Remark 3.19), hence $\frac{1}{2}m_2(X,x_0) < 2$ is equivalent to $m_2(X,x_0) = 3$. Also note that if $\tau(X,x_0) = 1$, $m_2(X,x_0) = 3$ and $\tau_2(X,x_0) = 1$, then $\frac{m_3(X,x_0)}{m_2(X,x_0)!} = \frac{1}{6}m_3(X,x_0)$ is well defined (see 3.22). Thus, the result follows from Lemmas 3.6, 3.20, and Propositions 3.17, 3.21, and 3.23. $\square$

Theorem 3.25. *A germ of a variety (X,x_0) of dimension d is a top singularity if and only if either*

(i) *there exist a minimal system of generators $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d+1}$ of the maximal ideal of $\widehat{\mathcal{O}_{X,x_0}}$ such that $\mathbf{x}_1\mathbf{x}_2 = 0$ or $\mathbf{x}_1^2 - \mathbf{x}_2^2\mathbf{x}_3 = 0,$ or*

(ii) *$d \geq 2$ and there exist a minimal system of generators $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$ of the maximal ideal of $\widehat{\mathcal{O}_{X,x_0}}$ such that one of the following holds:*
   (a) $\mathbf{x}_1^2 + \ldots + \mathbf{x}_\tau^2 + g(\mathbf{x}_{\tau+1}, \ldots, \mathbf{x}_{d+1}) = 0$ *where* $\tau \geq 2$, $g(x_{\tau+1}, \ldots, x_{d+1}) \in k[[x_{\tau+1}, \ldots, x_{d+1}]]$ *and* mult $g \geq 3.$

   (b) $\mathbf{x}_1^2 + \mathbf{x}_2^3 + p(\mathbf{x}_3, \ldots, \mathbf{x}_{d+1})\;\mathbf{x}_2 + q(\mathbf{x}_3, \ldots, \mathbf{x}_{d+1}) = 0$ *where* $p(x_3, \ldots, x_{d+1})$, $q(x_3, \ldots, x_{d+1}) \in k[[x_3, \ldots, x_{d+1}]]$, mult $p \geq 2$, mult $q \geq 3$, *and either* $2 \leq$ mult $p \leq 3$ *or* $3 \leq$ mult $q \leq 5.$

Proof: For $d = 1$ the statement is clear by Corollary 3.24. For $d \geq 2$, note that either (i) or (a) in the theorem holds if and only if (i) in Corollary 3.24 holds and that (b) in the theorem holds if and only if either (ii) or (iii) in Corollary 3.24 holds. $\square$

Theorem 3.26. *A germ of a variety (X,x_0) is a top singularity if and only if either*

(i) *(X,x_0) is a normal crossing double singularity or a pinch point or*

(ii) dim $X \geq 2$ *and (X,x_0) is a compound Du Val singularity*

Proof: Conditions (i) and (ii) imply that (X, x_0) is a top singularity by Propositions 3.11 and 3.14. The converse follows from the fact that a top singularity is a hypersurface double point and, under the classification of the defining equation according the invariants τ and m, a class of top singularities always satisfy condition either (i) or (ii) (Proposition 3.17, Lemma 3.20, Proposition 3.21, and Proposition 3.23). □

Remark 3.27. All the results and proofs in this section remain true if we replace completions by henselizations; that is, if we replace $\widehat{\mathcal{O}_{X,x_0}}$ by the henselization $\mathcal{O}^h_{X,x_0}$ of the local ring $\mathcal{O}_{X,x_0}$ and $k[[x_1, \ldots, x_N]]$ by $k\{x_1, \ldots, x_N\}$ for every $N \geq 1$. In fact, we have to apply the version of Weierstrass' preparation theorem for algebraic series and Hensel's Lemma in Propositions 3.21 and 3.23. In particular, Theorem 3.25 remains true if we replace $\widehat{\mathcal{O}_{X,x_0}}$ by $\mathcal{O}^h_{X,x_0}$ in both (i) and (ii) and we also replace $k[[x_{\tau+1}, \ldots, x_{d+1}]]$ (resp. $k[[x_3, \ldots, x_{d+1}]]$) by $k\{x_{\tau+1}, \ldots, x_{d+1}\}$ (resp. by $k\{x_3, \ldots, x_{d+1}\}$) in (a) (resp. (b)).

4. A variant of Shokurov's conjecture

In this section we consider the Mather version of Shokurov's second conjecture. Our main result is the following:

Theorem 4.1. *A pair (X, B) consisting of an arbitrary variety X and an effective $\mathbb{R}$-Cartier divisor B on X satisfies*

$$\dim X - 1 \leq \widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B)$$

if and only if either

(i) $B = 0$ *and (X, x) is a normal crossing double singularity or a pinch point,*
(ii) $B = 0$, $\dim X \geq 2$ *and (X, x) is a compound Du Val singularity or*
(iii) (X, x) *is non-singular and* $0 \leq \mathrm{mult}_x B \leq 1$.

In cases (i) *and* (ii)*, we have* $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X) = \dim X - 1$ *and in case* (iii)*, we have* $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B) = \mathrm{mld}(x; X, B) = \dim X - \mathrm{mult}_x B$ *and the minimal log discrepancy is computed by the exceptional divisor of the first blow-up at x.*

Proof: Let $d = \dim X$ and let (X, B) satisfy the condition $d-1 \leq \widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B)$ at a closed point $x \in X$. If (X, x) is singular, then by Proposition 1.2, we have $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X) \leq d - 1$ because $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X)$ is an integer (see 2.5). If $B \neq 0$ in a neighborhood of x, then

$$\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B) < \widehat{\mathrm{mld}}(x; X, \mathcal{J}_X) \leq d - 1,$$

in which case (X, B) does not satisfy the condition of the theorem. Therefore, if (X, x) is singular, then $B = 0$ and $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X) = d - 1$, *i.e.,* (X, x) is a top singularity. A top singularity is characterized in Theorem 3.26 as in (i) and (ii).

Hence, it is sufficient to characterize a pair (X, B) such that X is non-singular and $\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X B) = \mathrm{mld}(x; X, B) \geq d - 1$ in terms of (iii).

If $d = \dim X = 1$, then the statement is obvious since $\mathrm{mld}(x; X, B) = 1 - \mathrm{mult}_x B$.

Assume $d = \dim X \geq 2$ and (X, B) satisfies the inequality $\mathrm{mld}(x; X, B) \geq d - 1$, then the exceptional divisor E_1 of the blow-up $\varphi_1 : X_1 \to X$ of X at x should have the log discrepancy

$$k_{E_1} - \mathrm{ord}_{E_1} \varphi_1^* B + 1 \geq d - 1$$

(see 2.7 and also (9)). As $k_E = d - 1$ and $\mathrm{ord}_{E_1} \varphi_1^* B = \mathrm{mult}_x B$, this implies $\mathrm{mult}_x B \leq 1$.

Conversely, we assume (iii), that is $\mathrm{mult}_x B \leq 1$. Under this condition, we check the log discrepancy of every prime divisor over X with the center at x.

Let E be a prime divisor over X with the center at x, let $y \in E$ be the generic point, and let E appear in a resolution $f_0 : Y \to X$. Then, by Zariski's result (for example, see [Ko], VI, 1.3), we have a sequence of varieties $X_0, X_1, \dots, X_n$ and birational maps as follows:

> $X_0 = X$, $f_0 = f$.
> If $f_i : Y \dashrightarrow X_i$ is already defined, then let $Z_i \subset X_i$ be the closure of $p_i = f_i(y)$. Let $X_{i+1} = B_{Z_i}X_i$ and $f_{i+1} : Y \dashrightarrow X_{i+1}$ be the induced map. Then, the final birational map $f_n : Y \dashrightarrow X_n$ is isomorphic at y, *i.e.*, E appears on X_n.

Here $B_{Z_i}X_i$ is the blow-up of X_i with the center Z_i. Let $\varphi_i : X_i \to X_{i-1}$ be the blow-up morphism and $E_i \subset X_i$ be the exceptional divisor dominating Z_i. Note that the first blow-up $\varphi_1 : X_1 \to X_0 = X$ is at the closed point x because the center of E on X is x, whereas f_n is isomorphic at the generic points of E_n and E. We also note that X_i and E_i are non-singular at p_i for every $i = 1, \dots n$. Indeed, this is proved inductively. As X_1 is the blow-up at a closed point $x = p_0$, X_1 and E_1 are non-singular at every point. Suppose $i \geq 2$ and X_{i-1} and E_{i-1} are non-singular at p_{i-1}, then X_i is the blow-up with the non-singular center when one restricts the morphism on a neighborhood of p_{i-1}. As p_i is on the pull back of this neighborhood, X_i and E_i are non-singular at p_i.

Let $B^{(i)}$ be the strict transform of B on X_i, then from [Hi1] II sec. 5, Theorem 3 (p.233), we have

$$\operatorname{mult}_{p_i} B^{(i)} \leq \operatorname{mult}_{p_{i-1}} B^{(i-1)} \text{ for every } i = 1, \dots, n. \tag{25}$$

Let $a(E_i, X, B)$ be the discrepancy of (X, B) at the divisor E_i, *i.e.*,

$$a(E_i; X, B) = \operatorname{ord}_{E_i}(K_{X_i/X} - \Phi_i^*(B)),$$

where $\Phi_i : X_i \to X$ is the composite $\varphi_1 \circ \cdots \circ \varphi_i$. Note that the log discrepancy of (X, B) at the divisor E_i is $a(E_i; X, B) + 1$.

Claim. For every $i = 1, \dots, n$

$$a(E_i, X, B) \geq 0 \ \text{ and } \ a(E_i, X, B) \geq a(E_{i-1}, X, B).$$

By abuse of notation, we denote the strict transform of $E_{i-1} \subset X_{i-1}$ on X_j ($j \geq i$) by the same symbol E_{i-1}. Then, we have

$$\varphi_i^*(E_{i-1}) = E_{i-1} + E_i \tag{26}$$

by the non-singularity of X_{i-1} and E_{i-1} at p_{i-1} guaranteed in the discussion above.

Now we prove the claim by induction on i. First for $i = 1$, by substituting $K_{X_1/X} = (d-1)E_1$ and $\varphi_1^*(B) = (\operatorname{mult}_x B)E_1 + B^{(1)}$ into

$$a(E_1; X, B) = \operatorname{ord}_{E_1}(K_{X_1/X} - \varphi_1^*(B)),$$

we obtain

$$a(E_1; X, B) = (d-1) - \operatorname{mult}_x B \geq d - 2,$$

which is of course non-negative by our assumption $d \geq 2$.

Let $i \geq 2$ and assume that $a(E_j; X, B) \geq 0$ for all $j \leq i-1$ by induction hypothesis. Then

$$\begin{aligned} a(E_i; X, B) &= \operatorname{ord}_{E_i}(K_{X_i/X} - \Phi_i^*(B)) \\ &= \operatorname{ord}_{E_i}\left(K_{X_i/X_{i-1}} + \varphi_i^*\left(K_{X_{i-1}/X} - \Phi_{i-1}^*(B)\right)\right) \\ &= \operatorname{ord}_{E_i}\left(K_{X_i/X_{i-1}} + \varphi_i^*\Big(\sum_{j \leq i-1} a(E_j, X, B)E_j - B^{(i-1)}\Big)\right) \\ &\geq \operatorname{ord}_{E_i}(K_{X_i/X_{i-1}}) + a(E_{i-1}; X, B) - \operatorname{mult}_{p_{i-1}} B^{(i-1)}. \end{aligned}$$

Here we used (26) and the hypothesis of the induction. We may assume that $\operatorname{codim}\overline{\{p_{i-1}\}} \geq 2$, because if $\operatorname{codim}\overline{\{p_{i-1}\}} = 1$, then f_{i-1} is already isomorphic at the generic point $y \in E$. (We may assume that n is taken to be minimal.) As $\operatorname{ord}_{E_i}(K_{X_i/X_{i-1}}) = \operatorname{codim}\overline{\{p_{i-1}\}} - 1 \geq 1$ and $\operatorname{mult}_{p_{i-1}} B^{(i-1)} \leq \operatorname{mult}_x B \leq 1$ by (25), we obtain

$$a(E_i, X, B) \geq a(E_{i-1}, X, B) \geq 0$$

as claimed.

From this, the log discrepancy at E is $a(E; X, B) + 1 = a(E_n; X, B) + 1 \geq a(E_1; X, B) + 1 = d - \operatorname{mult}_x B \geq d - 1$. Therefore, the inequality $\dim X - 1 \leq \widehat{\operatorname{mld}}(x; X, \mathcal{J}_X B) = \operatorname{mld}(x; X, B)$ holds and the minimal log discrepancy $d - \operatorname{mult}_x B$ is computed by the exceptional divisor of the first blow-up at x. □

As a corollary of the theorem, we have the "if" part of Conjecture 1.3:

Corollary 4.2. *The inequality*

$$\dim X - 1 < \operatorname{mld}(x; X, B)$$

holds if (X, x) is non-singular and $\operatorname{mult}_x B < 1$. In this case, the minimal log discrepancy is computed by the exceptional divisor of the first blow-up at x.

Since $\operatorname{mld}(x; X, B) = \widehat{\operatorname{mld}}(x; X, \mathcal{J}_X B)$ for (X, x), which is a normal complete intersection by (9), we have the following statement for usual mld as a further corollary.

Corollary 4.3. *A pair (X, B) consisting of a normal complete intersection variety X at a closed point x and an effective $\mathbb{R}$-Cartier divisor B on X satisfies*

$$\dim X - 1 \leq \operatorname{mld}(x; X, B)$$

if and only if either (i) *or* (ii) *or* (iii) *in the theorem holds.*

By this and Proposition 2.10, Corollary 1.6 follows.

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Shihoko Ishii,
Graduate School of Mathematical Science, University of Tokyo,
3-8-1 Komaba, Meguro, 153-8914 Tokyo, Japan.
E-mail: shihoko@ms.u-tokyo.ac.jp

Ana J. Reguera,
Dep. de Álgebra, Geometría y Topología, Universidad de Valladolid,
Paseo Belén 7, 47011 Valladolid, Spain.
E-mail: areguera@agt.uva.es